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2006 J. Phys. A: Math. Gen. 39 L27

(http://iopscience.iop.org/0305-4470/39/1/L04)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) L27–L34 doi:10.1088/0305-4470/39/1/L04

LETTER TO THE EDITOR

Exact solutions for discrete and ultradiscrete modified KdV equations and their relation to box-ball systems

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Received 28 October 2005
Published 7 December 2005
Online at stacks.iop.org/JPhysA/39/L27

Abstract
A new class of solutions is proposed for discrete and ultradiscrete modified KdV equations. These are directly related to solutions of the box and ball system with a carrier. Moreover, an extended box and ball system and its exact solutions are discussed.

PACS numbers: 02.30.Ik, 05.45.Yv, 87.17.−d

1. Introduction

The modified Korteweg–de Vries (mKdV) equation,

$$\frac{\partial w}{\partial t} - w^2\frac{\partial w}{\partial x} + \frac{\partial^3 w}{\partial x^3} = 0 \tag{1}$$

is one of the famous soliton equations. A discrete analogue of this equation was first proposed by Hirota as the Bäcklund transformation for the discrete KdV equation [1]. He also showed that the discrete mKdV equation is a reduced case of the Hirota–Miwa equation [2]. An extended version of the discrete mKdV equation, which has an extra parameter, was introduced by Tsujimoto and Hirota in order to discuss the existence of higher order conserved quantities [3]. As for exact solutions, Maruno *et al* presented an N-soliton solution and algebraic solutions for the discrete mKdV equation using its bilinear form [4].

Ultradiscrete equations are evolution systems in which both dependent and independent variables take discrete values [5, 6]. From both theoretical and applied points of view, it is an interesting problem to construct ultradiscrete systems from discrete equations. In [7], we have given an ultradiscrete analogue of the sine-Gordon equation and discussed its soliton solutions. For the mKdV equation, Takahashi and Matsukidaira [8] have shown that an ultradiscrete system corresponding to the extended version of the discrete mKdV equation reduces to 'box and ball system with a carrier (BBSC)'. Although they give some exact solutions of the BBSC, the relation between them and solutions of the discrete mKdV equation is not yet clear.

In this letter, we give a class of solutions for the extended version of the discrete mKdV equation and show that they directly relate to those of the BBSC. We also propose an extended BBSC in which capacities of the boxes and the carrier take negative values. In section 2, we first present a class of solutions of the extended mKdV equation. Then we introduce a new dependent variable which is suitable for discussing the relation to the BBSC. In section 3, we ultradiscretize the extended mKdV equation and its solutions. Then in section 4, we derive an ultradiscrete system for the new variable and show that the system and its solutions are directly related to those of the BBSC. We also discuss an extension of the BBSC in this section. Finally in section 5 we give concluding remarks.

2. Solutions of the discrete mKdV equation

The extended version of the discrete mKdV equation [3] may be written as

$$w_n^{m+1}\frac{1+a^{-1}dw_{n+1}^m}{1+a^{-1}d^{-1}w_{n+1}^m} = w_n^{m-1}\frac{1+a^{-1}dw_{n-1}^m}{1+a^{-1}d^{-1}w_{n-1}^m}, \tag{2}$$

where a and d are parameters related to the mesh sizes for m and n. If we replace w_n^m, a, d and m with $1+(w_n-1)/[\alpha(w_n+1)]$, $(1-\alpha)/(1+\alpha)$, $(1+\delta)/(1-\delta)$ and s/δ, respectively and take the limit $\delta \to 0$, we have from (2)

$$\frac{\partial w_n}{\partial s} + \frac{(1-\alpha^2)[1-\alpha^2(w_n-1)^2](w_{n+1}-w_{n-1})}{2[1+\alpha^2(w_{n+1}-1)][1+\alpha^2(w_{n-1}-1)]} = 0. \tag{3}$$

If we further replace s by $6t/[\alpha^3(1-\alpha^2)]$ and n by $x/\alpha + 6(1+\alpha^2)t/\alpha^3$ and take the limit $\alpha \to 0$, then (3) reduces to (1).

By introducing a variable transformation

$$w_n^m = \frac{f_{n+1}^m g_{n-1}^m}{f_{n-1}^m g_{n+1}^m} \tag{4}$$

in (2) and decoupling the resulting equation, we have the bilinear form

$$\begin{aligned}(1+a^{-1}d^{-1}) f_n^{m+1} g_n^{m-1} &= f_{n-1}^m g_{n+1}^m + a^{-1}d^{-1} f_{n+1}^m g_{n-1}^m \\ (1+a^{-1}d) f_n^{m-1} g_n^{m+1} &= f_{n-1}^m g_{n+1}^m + a^{-1}d f_{n+1}^m g_{n-1}^m.\end{aligned} \tag{5}$$

If we take $a = 1$, then (5) becomes the bilinear form of the discrete mKdV equation proposed by Hirota [1, 2].

The N-soliton solution of (2) is obtained from f_n^m and g_n^m, in terms of polynomials of exponential functions. Since the N-soliton solution given by Maruno *et al* [4] has negative signs in the polynomials, difficulties arise when we apply the procedure of ultradiscretization. We here give another class of N-soliton solution which is suitable for the ultradiscretization.

Let $p_j, \omega_j, c_j (j = 1, 2, \ldots, N)$ be parameters satisfying

$$\begin{aligned}
\exp(2p_j) &= [\exp(2c_j)(1 + ad^{-1}) + 1 + a^{-1}d^{-1}] \\
&\quad \times [\exp(2c_j)(1 + ad) + 1 + a^{-1}d] \\
&\quad \times [\exp(2c_j)(1 + a^{-1}d^{-1}) + 1 + ad^{-1}]^{-1} \\
&\quad \times [\exp(2c_j)(1 + a^{-1}d) + 1 + ad]^{-1} \\
\exp(2\omega_j) &= [\exp(2c_j)(1 + ad^{-1}) + 1 + a^{-1}d^{-1}] \\
&\quad \times [\exp(2c_j)(1 + a^{-1}d) + 1 + ad] \\
&\quad \times [\exp(2c_j)(1 + a^{-1}d^{-1}) + 1 + ad^{-1}]^{-1} \\
&\quad \times [\exp(2c_j)(1 + ad) + 1 + a^{-1}d]^{-1}.
\end{aligned} \tag{6}$$

If we eliminate c_j, we have the dispersion relation

$$(1 + ad)(1 + a^{-1}d) \sinh \omega_j - (1 + d)(1 - d) \sinh p_j = 0. \tag{7}$$

Moreover, we define phase functions η_j and an interaction factor $\exp(A_{jk})$ by

$$\eta_j = p_j n + \omega_j m + \eta_j^{(0)} \tag{8}$$

$$\exp(A_{jk}) = \frac{\sinh^2(c_j - c_k)}{\sinh^2(c_j + c_k)}, \tag{9}$$

where $\eta_j^{(0)}$ is an arbitrary phase constant. Then the N-soliton solution we propose is written as

$$\begin{aligned}
f_n^m &= \sum_{\mu} \exp\left[\sum_{j=1}^{N} \mu_j(\eta_j + c_j) + \sum_{j<k}^{N} \mu_j \mu_k A_{jk}\right] \\
g_n^m &= \sum_{\mu} \exp\left[\sum_{j=1}^{N} \mu_j(\eta_j - c_j) + \sum_{j<k}^{N} \mu_j \mu_k A_{jk}\right],
\end{aligned} \tag{10}$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$, $\mu_j \in \{0, 1\}$. In particular, the one-soliton solution is given by

$$f_n^m = 1 + \exp(\eta_1 + c_1) \qquad g_n^m = 1 + \exp(\eta_1 - c_1) \tag{11}$$

and the two-soliton solution by

$$\begin{aligned}
f_n^m &= 1 + \exp(\eta_1 + c_1) + \exp(\eta_2 + c_2) + \exp(\eta_1 + \eta_2 + c_1 + c_2 + A_{12}) \\
g_n^m &= 1 + \exp(\eta_1 - c_1) + \exp(\eta_2 - c_2) + \exp(\eta_1 + \eta_2 - c_1 - c_2 + A_{12}).
\end{aligned} \tag{12}$$

For the variable w_n the one-soliton solution is written as

$$w_n^m = \frac{\cosh \eta_1 + \cosh(p_1 + c_1)}{\cosh \eta_1 + \cosh(p_1 - c_1)}. \tag{13}$$

It is to be noted that $w_n^m \to 1$ as $|n| \to \infty$ and that the amplitude

$$\frac{\cosh^2(p_1 + c_1)/2}{\cosh^2(p_1 - c_1)/2} - 1 \tag{14}$$

depends on the values of a and d.

Let us introduce a new dependent variable u_n^m by

$$u_n^m = \frac{f_n^m g_{n-1}^{m-1}}{f_{n-1}^{m-1} g_n^m}. \tag{15}$$

Then from (5), we find that u_n^m satisfies

$$u_n^{m+1} = \frac{1+a^{-1}d}{1+a^{-1}d^{-1}} \frac{1+a^{-1}d^{-1}\prod_{i=-\infty}^{0}\left(u_{n+1+i}^{m+i}/u_{n-1+i}^{m+i}\right)}{a^{-1}du_{n+1}^m + \prod_{i=-\infty}^{0}\left(u_{n-1+i}^{m+i}/u_{n+i}^{m-1+i}\right)}. \tag{16}$$

We shall see in section 4 that the ultradiscrete limit of (16) gives the BBSC discussed by Takahashi and Matsukidaira [8]. The one-soliton solution for u_n^m is written as

$$u_n^m = \frac{\cosh\left(\eta_1 - \frac{p_1+\omega_1}{2}\right) + \cosh\left(-c_1 - \frac{p_1+\omega_1}{2}\right)}{\cosh\left(\eta_1 - \frac{p_1+\omega_1}{2}\right) + \cosh\left(c_1 - \frac{p_1+\omega_1}{2}\right)}. \tag{17}$$

The behaviour of u_n^m is almost the same as w_n^m.

3. Ultradiscrete mKdV equation

We now apply the procedure of ultradiscretization to (2) and (5). Replacing a, d, w_n^m, f_n^m and g_n^m with

$$\begin{gathered} a = \exp\left(\frac{A}{\varepsilon}\right), \qquad d = \exp\left(\frac{D}{\varepsilon}\right), \qquad w_n^m = \exp\left(\frac{W_n^m}{\varepsilon}\right) \\ f_n^m = \exp\left(\frac{F_n^m}{\varepsilon}\right), \qquad g_n^m = \exp\left(\frac{G_n^m}{\varepsilon}\right), \end{gathered} \tag{18}$$

respectively, and taking the limit of $\varepsilon \to +0$, we have an ultradiscrete system for W_n^m,

$$\begin{aligned} &W_n^{m+1} + \max\left(0, W_{n+1}^m - A + D\right) - \max\left(0, W_{n+1}^m - A - D\right) \\ &\quad = W_n^{m-1} + \max\left(0, W_{n-1}^m - A + D\right) - \max\left(0, W_{n-1}^m - A - D\right), \end{aligned} \tag{19}$$

and one for F_n^m and G_n^m,

$$\begin{aligned} F_n^{m+1} + G_n^{m-1} &= \max\left[F_{n-1}^m + G_{n+1}^m - \max(0, -A-D), F_{n+1}^m + G_{n-1}^m - \max(0, A+D)\right] \\ F_n^{m-1} + G_n^{m+1} &= \max\left[F_{n-1}^m + G_{n+1}^m - \max(0, -A+D), F_{n+1}^m + G_{n-1}^m - \max(0, A-D)\right]. \end{aligned} \tag{20}$$

The dependent variable W_n^m is written in terms of F_n^m, G_n^m by

$$W_n^m = F_{n+1}^m + G_{n-1}^m - F_{n-1}^m - G_{n+1}^m. \tag{21}$$

Equation (19) admits exact solutions which are obtained from the ultradiscrete analogue of the N-soliton solution (10) through (21). In order to set the analogue, we first replace the parameters in (10) by $p_j = P_j/\varepsilon, \omega_j = \Omega_j/\varepsilon, c_j = C_j/\varepsilon$ and $\eta_j^{(0)} = \Xi_j^{(0)}/\varepsilon$. Then by taking the limit of $\varepsilon \to +0$, we find that (6) and (7) reduce to

$$\begin{aligned} 2P_j &= \max[2C_j + \max(0, A-D), \max(0, -A-D)] \\ &\quad + \max[2C_j + \max(0, A+D), \max(0, -A+D)] \\ &\quad - \max[2C_j + \max(0, -A-D), \max(0, A-D)] \\ &\quad - \max[2C_j + \max(0, -A+D), \max(0, A+D)] \\ 2\Omega_j &= \max[2C_j + \max(0, A-D), \max(0, -A-D)] \\ &\quad + \max[2C_j + \max(0, -A+D), \max(0, A+D)] \\ &\quad - \max[2C_j + \max(0, -A-D), \max(0, A-D)] \\ &\quad - \max[2C_j + \max(0, A+D), \max(0, -A+D)] \end{aligned} \tag{22}$$

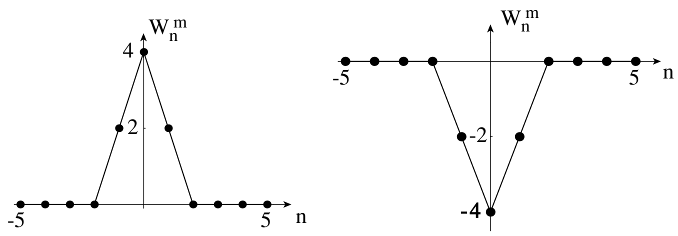

Figure 1. Examples of triangular pulses. The left is for the case $C_1 = 2$, $A = 2$, $D = 1$ and the right for the case $C_1 = 2$, $A = -2$, $D = 1$.

and

$$\max[|P_j + D|, \max(|A|, |D|) + \Omega_j] = \max[|P_j - D|, \max(|A|, |D|) - \Omega_j], \tag{23}$$

respectively. Similarly, the ultradiscrete limit of the function η_j and the interaction factor $\exp(A_{jk})$ are given by

$$\Xi_j = P_j n + \Omega_j m + \Xi_j^{(0)} \tag{24}$$

and

$$2(|C_j - C_k| - |C_j + C_k|), \tag{25}$$

respectively.

Employing these variables, the ultradiscrete analogue of the one-soliton solution (11) is written as

$$F_n^m = \max(0, \Xi_1 + C_1), \qquad G_n^m = \max(0, \Xi_1 - C_1) \tag{26}$$

and

$$W_n^m = \max(|\Xi_1|, |P_1 + C_1|) - \max(|\Xi_1|, |P_1 - C_1|). \tag{27}$$

This solution describes a triangular pulse, the sign of which depends on the parameters A and D (see Figure 1).

The ultradiscrete analogue of the two-soliton solution is given by

$$\begin{aligned} F_n^m &= \max[0, \Xi_1 + C_1, \Xi_2 + C_2, \Xi_1 + \Xi_2 + C_1 + C_2 + 2(|C_1 - C_2| - |C_1 + C_2|)] \\ G_n^m &= \max[0, \Xi_1 - C_1, \Xi_2 - C_2, \Xi_1 + \Xi_2 - C_1 - C_2 + 2(|C_1 - C_2| - |C_1 + C_2|)]. \end{aligned} \tag{28}$$

Figure 2 shows the behaviour of W_n^m obtained from (28). The interaction of two triangular pulses is similar to that described by the two-soliton solution for the mKdV equation.

It is to be noted that the ultradiscrete analogue of the N-soliton solution is obtained by applying the same procedure to (10).

4. Relation to the box and ball system with a carrier

In order to obtain an ultradiscrete system corresponding to the BBSC, we first replace a, d and u_n^m in (16) with

$$a = \exp\left(\frac{A}{\varepsilon}\right), \qquad d = \exp\left(\frac{D}{\varepsilon}\right), \qquad u_n^m = \exp\left(\frac{U_n^m}{\varepsilon}\right), \tag{29}$$

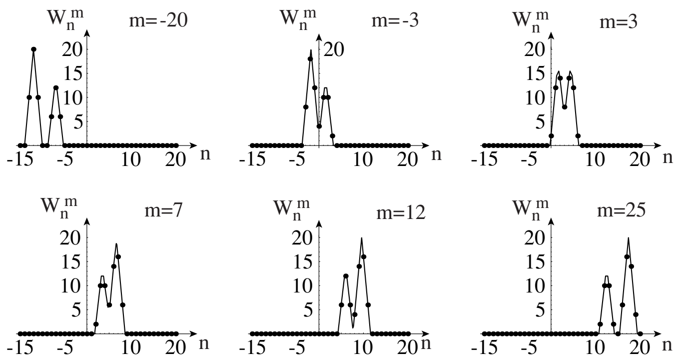

Figure 2. An example of interaction of two pulses in the ultradiscrete system. The parameters are $A = 10$, $D = 6$, $(C_1, \Xi_1^{(0)}) = (6, 0)$ and $(C_2, \Xi_2^{(0)}) = (10, 0)$.

respectively. Then taking the limit of $\varepsilon \to +0$, we have an ultradiscrete system for U_n^m,

$$U_n^{m+1} = \min\left[\max(0, A - D) - U_{n+1}^m, \sum_{j=-\infty}^{0} U_{n+j}^{m-1+j} - \sum_{j=-\infty}^{0} U_{n-1+j}^{m+j} + \max(0, -A + D)\right] + \max\left[-\max(0, -A - D), \sum_{j=-\infty}^{0} U_{n+1+j}^{m+j} - \sum_{j=-\infty}^{0} U_{n-1+j}^{m+j} - \max(0, A + D)\right]. \tag{30}$$

The dependent variable U_n^m is written in terms of F_n^m, G_n^m by

$$U_n^m = F_n^m + G_{n-1}^{m-1} - F_{n-1}^{m-1} - G_n^m. \tag{31}$$

Replacing the parameters and the coordinates in the ultradiscrete system for U_n^m (30) by

$$L = A - D, \qquad M = A + D, \qquad j = \frac{n+m}{2}, \qquad t = \frac{m-n}{2}, \tag{32}$$

we have

$$U_j^{t+1} = \min\left[\max(0, L) - U_j^t, \sum_{i=-\infty}^{j-1} U_i^t - \sum_{i=-\infty}^{j-1} U_i^{t+1} + \max(0, -L)\right] + \max\left[-\max(0, -M), \sum_{i=-\infty}^{j} U_i^t - \sum_{i=-\infty}^{j-1} U_i^{t+1} - \max(0, M)\right], \tag{33}$$

which is nothing but the BBSC in the case of $0 < L < M$ [8]. In the BBSC, L and M correspond to the capacities of the boxes and carrier, respectively.

In the preceding section, we have discussed the ultradiscrete limit of F_n^m and G_n^m corresponding to the N-soliton solution of the discrete mKdV equation. Employing this result, we can construct exact solutions for (33) by means of (31). In the case of $0 < L < M$, they are equivalent to those presented in [8], which shows a direct relationship between solutions of the mKdV equation and those of the BBSC. This relationship was never made explicit until now.

However, the parameters A and D are arbitrary in our treatment. Hence the capacities L and M defined in (32) are also arbitrary, which means we can extend the BBSC to a system with negative 'capacities' of boxes and/or a carrier.

$t = 0$	0	−1	−4	−4	−4	−4	−2	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	−2	−4	−4	−4	−4	−1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	−3	−4	−4	−4	−4	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	−4	−4	−4	−4	−3	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	−1	−4	−4	−4	−4	−2	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	−2	−4	−4	−4	−4	−1	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	−3	−4	−4	−4	−4	0

$\longrightarrow j$

Figure 3. A pulse with negative amplitude: $(C_1, L, M) = (19/2, -7, -4)$.

$t = 0$	0	1	4	4	4	4	2	0	0	0	0	0	0	0	0	0	0	0
1	0	1	4	4	4	4	2	0	0	0	0	0	0	0	0	0	0	0
2	0	1	4	4	4	4	2	0	0	0	0	0	0	0	0	0	0	0
3	0	1	4	4	4	4	2	0	0	0	0	0	0	0	0	0	0	0
4	0	1	4	4	4	4	2	0	0	0	0	0	0	0	0	0	0	0
5	0	1	4	4	4	4	2	0	0	0	0	0	0	0	0	0	0	0
6	0	1	4	4	4	4	2	0	0	0	0	0	0	0	0	0	0	0

$\longrightarrow j$

Figure 4. A standing wave solution: $(C_1, L, M) = (19/2, 7, -3)$.

Solutions of such a system immediately follow from the results we have given so far. We here present two examples of solutions corresponding to the one-soliton solution (26). Figure 3 shows an example for $L < M < 0$. A pulse with negative amplitude propagates to the right. If we change the sign of the values in all boxes, we recover the corresponding solution for $0 < L < M$. Figure 4 is an example for $M < 0 < L$. In this case we have a standing wave solution, which is not obtained in the original BBSC.

5. Concluding remarks

We have given a new class of soliton solutions for the discrete mKdV equation and we ultradiscretized both the equation and the solutions. Moreover, by introducing a new dependent variable, we have shown that the ultradiscrete system also directly relates to the BBSC and we presented an extended BBSC in which the capacities of boxes and carrier can have negative values.

The new class of soliton solutions considered in this letter has the peculiar property that the sign of the solutions is completely determined by the parameters included in the equation. The continuous mKdV equation admits exact solutions describing soliton–anti-soliton interactions, which are not covered by our solutions. As mentioned in section 2, we have difficulty applying the procedure of ultradiscretization to solutions with non-definite sign. Hence, in order to treat such solutions we have to introduce a different method of ultradiscretization, on which we shall report in a forthcoming paper.

Acknowledgments

This work was partially supported by the Japan Society for the Promotion of Science (JSPS). The authors also thank Professor Ralph Willox for helpful comments.

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